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TEMPERATURE FIELDS IN A TWO-LAYERED PLATE WITH A SEMI-INFINITE SLIT ALONG THE INTERFACE

B. V. Nerubailo and L. G. Smirnov

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In this paper, we study steady temperature fields in a two-layered plate containing a semi-infinite slit along the interface. It is assumed that the heat conduction coefficient is constant at the boundary and that outside of the slit the contact is ideal. To obtain a solution we use an analog of the Wiener–Hopf method. Calculations are illustrated by curves of temperature field distribution along the sides of the slit.

We assume that the heat-conducting layers ($0 < y < h_1$, $-\infty < x < +\infty$) and ($-h_2 < y < 0$, $-\infty < x < 0$) are in ideal contact when $x > 0$ and are thermally insulated from one another when $x < 0$.

The heat conduction equations for the layers are written as

$$\Delta T_j = \frac{\partial^2 T_j}{\partial x^2} + \frac{\partial^2 T_j}{\partial y^2} = 0 \quad (j = 1, 2); \tag{1}$$

the boundary conditions are

$$\alpha_j \frac{\partial T_j}{\partial y}\bigg|_{y=y_j} = (\beta_j T_j + \gamma_j)\bigg|_{y=y_j} \quad (j = 1, 2;\ y_1 = h_1,\ y_2 = -h_2), \tag{2}$$

and the conjugation conditions at the common boundary of the layers are

$$\lambda_1 \frac{\partial T_1}{\partial y} = \lambda_2 \frac{\partial T_2}{\partial y} = 0 \quad (y = 0,\ x < 0); \tag{3}$$

$$T_1 = T_2, \quad \lambda_1 \frac{\partial T_1}{\partial y} = \lambda_2 \frac{\partial T_2}{\partial y} \quad (y = 0,\ x > 0). \tag{4}$$

Here λ_j is the heat conduction coefficient of the material of the first and second layers, respectively ($j = 1, 2$) and α_j, β_j, and γ_j are constants ($j = 1, 2$) ($\alpha_j = 0$ when the boundary condition is of the first kind and $\beta_j \neq 0$). We seek a solution $T_j(x, y)$ in the form

$$T_j(x, y) = T_j^*(x, y) + T_j^{(0)}(x, y), \tag{5}$$

where $T_j^{(0)}(x, y)$ is a solution of Eq. (1) subject to the boundary conditions (2) and conjugation condition (4) which are satisfied along the whole line ($-\infty < x < +\infty$).

The solution $T_j^{(0)}(x, y)$ is

$$T_j^{(0)}(x, y) = a_j y + b_j \quad (j = 1, 2).$$

Here

$$a_1 = (\gamma_1\beta_2 - \gamma_2\beta_1)/(\alpha_1\beta_2 - \beta_1\beta_2 h_1 - \alpha_2\beta_1\lambda_1/\lambda_2 - \beta_1\beta_2\lambda_1 h_2/\lambda_2),$$

$$b_1 = ((\alpha_1 - \beta_1 h_1)a_1 - \gamma_1)/\beta_1, \quad a_2 = (\lambda_1/\lambda_2)a_1, \quad b_2 = b_1.$$

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Conditions (1)–(4) with account of (5) can then be written as ($q = a_1\lambda_1 = a_2\lambda_2$)

$$\alpha_j \frac{\partial T_j^*}{\partial y}\bigg|_{y=y_j} = \beta_j T_j^*\bigg|_{y=y_j} \quad (j = 1, 2); \tag{6}$$

$$\frac{\partial T_1^*}{\partial y}\bigg|_{y=0} = -\frac{q}{\lambda_1}, \quad \frac{\partial T_2^*}{\partial y}\bigg|_{y=0} = -\frac{q}{\lambda_2} \quad (x < 0); \tag{7}$$

$$T_1^*\bigg|_{y=0} = T_2^*\bigg|_{y=0}, \qquad \lambda_1 \frac{\partial T_1^*}{\partial y}\bigg|_{y=0} = \lambda_2 \frac{\partial T_2^*}{\partial y}\bigg|_{y=0} \quad (x > 0). \tag{8}$$

We will seek $T_j^*(x, y)$ in the form [1]

$$T_j^*(x, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(A_j(\xi) e^{i\xi y} + B_j(\xi) e^{-i\xi y}\right) e^{\xi x}\, d\xi, \tag{9}$$

where $A_j(\xi)$ and $B_j(\xi)$ are unknown functions ($j = 1, 2$). Using (9), from conditions (6), we obtain

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(i\alpha_j \xi \left(A_j(\xi) e^{i\xi y_j} - B_j(\xi) e^{-i\xi y_j}\right) - \beta_j \left(A_j(\xi) e^{i\xi y_j} + B_j(\xi) e^{-i\xi y_j}\right)\right) e^{\xi x}\, d\xi = 0,$$

whence

$$B_j(\xi) = e^{2i\xi y_j}(i\alpha_j\xi - \beta_j)/(i\alpha_j\xi + \beta_j) A_j(\xi) \quad (j = 1, 2).$$

We now represent conditions (7) and (8) as

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} X_1(\xi) A_1(\xi) e^{\xi x}\, d\xi = -\frac{q}{\lambda_1}, \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} X_2(\xi) A_2(\xi) e^{\xi x}\, d\xi = -\frac{q}{\lambda_2}; \tag{10}$$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Y_1(\xi) A_1(\xi) e^{\xi x}\, d\xi = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Y_2(\xi) A_2(\xi) e^{\xi x}\, d\xi, \quad \frac{\lambda_1}{2\pi i} \int_{-i\infty}^{i\infty} X_1(\xi) A_1(\xi) e^{\xi x}\, d\xi = \frac{\lambda_2}{2\pi i} \int_{-i\infty}^{i\infty} X_2(\xi) A_2(\xi) e^{\xi x}\, d\xi. \tag{11}$$

Here

$$X_j(\xi) = i\xi \left(1 - \frac{i\alpha_j\xi - \beta_j}{i\alpha_j\xi + \beta_j} e^{2i\xi y_j}\right); \quad Y_j(\xi) = 1 + \frac{i\alpha_j\xi - \beta_j}{i\alpha_j\xi + \beta_j} e^{2i\xi y_j} \quad (j = 1, 2).$$

Setting $A_j^*(\xi) = X_j(\xi) A_j(\xi)$, instead of (10) and (11) we have

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} A_1^*(\xi) e^{\xi x}\, d\xi = -\frac{q}{\lambda_1} \quad (x < 0); \tag{12}$$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} A_2^*(\xi) e^{\xi x}\, d\xi = -\frac{q}{\lambda_2} \quad (x < 0); \tag{13}$$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Y_1(\xi) X_1^{-1}(\xi) A_1^*(\xi) e^{\xi x}\, d\xi = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Y_2(\xi) X_2^{-1}(\xi) A_2^*(\xi) e^{\xi x}\, d\xi \quad (x > 0); \tag{14}$$

$$\frac{\lambda_1}{2\pi i} \int_{-i\infty}^{i\infty} A_1^*(\xi) e^{\xi x}\, d\xi = \frac{\lambda_2}{2\pi i} \int_{-i\infty}^{i\infty} A_2^*(\xi) e^{\xi x}\, d\xi \quad (x > 0). \tag{15}$$

If we now set $A_2^*(\xi) = \lambda_1/\lambda_2 A_1^*(\xi)$, condition (15) is satisfied automatically and conditions (12) and (13) reduces to one. As a result, we obtain the following two conditions:

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} A_1^*(\xi)e^{\xi x}\,d\xi = -\frac{q}{\lambda_1} \quad (x<0); \tag{16}$$

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} F(\xi)A_1^*(\xi)e^{\xi x}\,d\xi = 0 \quad (x>0), \tag{17}$$

where

$$\begin{gathered} F(\xi) = F_0(\xi)/(F_1(\xi)F_2(\xi)), \\ F_0(\xi) = [(\alpha_1\xi\cos(\xi y_1) - \beta_1\sin(\xi y_1))(\alpha_2\xi\sin(\xi y_2) + \beta_2\cos(\xi y_2)) \\ -\gamma(\alpha_1\xi\sin(\xi y_1) + \beta_2\cos(\xi y_1))(\alpha_2\xi\cos(\xi y_2) - \beta_2\sin(\xi y_2))]/\xi, \\ F_1(\xi) = \alpha_1\xi\sin(\xi y_1) + \beta_1\cos(\xi y_1), \quad F_2(\xi) = \alpha_2\xi\sin(\xi y_2) + \beta_2\cos(\xi y_2). \end{gathered} \tag{18}$$

The functions $F_j(\xi)$ $(j=0,1,2)$ are entire functions of the first order [1]; moreover, each of them is an even function of ξ. Hence, the Weierstrass representation for each of them according to Hadamard's theorem [2] has the form

$$f(\xi) = e^b \prod_{m=1}^{\infty}\left(1 - \xi^2/\delta_m^2\right).$$

Here b is a constant and δ_m are zeros of the function $f(\xi)$ $(m = 1, 2, \dots, \infty)$.

The function $F(\xi)$ can be written in the form

$$F(\xi) = F^+(\xi)F^-(\xi), \tag{19}$$

where

$$\begin{gathered} F^+(\xi) = \frac{F_0^+(\xi)}{F_1^+(\xi)F_2^+(\xi)} = g(\xi)\prod_{m=1}^{\infty}\left(1-\xi/a_{m0}^+\right)\Big/\left(\prod_{m=1}^{\infty}\left(1-\xi/a_{m1}^+\right)\prod_{m=1}^{\infty}\left(1-\xi/a_{m2}^+\right)\right), \\ F^-(\xi) = \frac{F_0^-(\xi)}{F_1^-(\xi)F_2^-(\xi)} = \prod_{m=1}^{\infty}\left(1-\xi/a_{m0}^-\right)\Big/\left(\prod_{m=1}^{\infty}\left(1-\xi/a_{m1}^-\right)\prod_{m=1}^{\infty}\left(1-\xi/a_{m2}^-\right)\right); \end{gathered} \tag{20}$$

$a_{mj}^{\pm}$ are zeros of the functions $F_j(\xi)$ lying on the right-hand and left-hand side of the complex plane, respectively $(j = 0,1,2$ and $m = 1,2,\dots,\infty)$, and $g(\xi)$ is an entire function without zeros in the whole complex plane.

We set $A_j^*(\xi) = a/(\xi F^-(\xi))$ (a is an unknown constant) and substitute this into expression (17). As a result, we have

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} aF(\xi)/(\xi F^-(\xi))e^{\xi x}\,d\xi = \frac{a}{2\pi i}\int\limits_{-i\infty}^{i\infty} F^+(\xi)/\xi e^{\xi x}\,d\xi \quad (x>0). \tag{21}$$

For $x > 0$ in the region $\operatorname{Re}\xi < 0$, the holomorphic function $F^+(\xi)/\xi$ has no poles and satisfies the conditions of Jordan's lemma [1]. Indeed, in the region $\operatorname{Re}\xi < 0$, as $|\xi| \to \infty$, the asymptotic formula

$$F(\xi) = F^+(\xi)F^-(\xi) \sim (1-\gamma)$$

is valid, whence, taking into account (19) and (20), we infer that

$$F^+(\xi) \sim F^-(\xi) \sim \sqrt{(1-\gamma)} = \text{const},$$

as $|\xi| \to \infty$ $(\operatorname{Re}\xi < 0)$ and hence $\lim\limits_{|\xi|\to\infty}(F^+(\xi)/\xi) = 0$ and the integral appearing in (21) vanishes. Then, substituting $A_1^*(\xi) = a/(F^-(\xi)\xi)$ into (16), we obtain

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} a/(\xi F^-(\xi))e^{\xi x}\,d\xi = -\frac{q}{\lambda_1} \quad (x < 0).$$

Hence, since the function also satisfies the conditions of Jordan's lemma for $\mathrm{Re}\,\xi \geqslant 0$ and has a single pole of the first order at $\xi = 0$, we have $a = q/\lambda_1$. Bearing (10) in mind, we write the expression for $T_j^*(x,y)$:

$$T_j^*(x,y) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\Big(A_j(\xi)e^{i\xi y} + B_j(\xi)e^{-i\xi y}\Big)e^{\xi x}\,d\xi$$

$$= \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\Big(e^{i\xi y} + e^{-i\xi y + 2i\xi y_j}(i\alpha_j - \beta_j)/(i\alpha_j + \beta_j)\Big)\Big/X_j(\xi)A_j^*(\xi)e^{\xi x}\,d\xi$$

$$= \frac{(q/\lambda_j)}{2\pi i}\int\limits_{-i\infty}^{i\infty}\Big(\alpha_j\xi\cos(\xi(y - y_j)) + \beta_j\sin(\xi(y-y_j))\Big)\Big/\Big(\xi^2F_j(\xi)F^-(\xi)\Big)e^{\xi x}\,d\xi$$

$$= \frac{(q/\lambda_j)}{2\pi i}\int\limits_{-i\infty}^{i\infty}\Big(\alpha_j\xi\cos(\xi(y - y_j)) + \beta_j\sin(\xi(y-y_j))\Big)F_k^-(\xi)\Big/\Big(\xi^2F_j^+(\xi)F_0^-(\xi)\Big)e^{\xi x}\,d\xi$$

($k = 1$, if $j = 2$ and $k = 2$, if $j = 1$).

The functions $F_j(\xi)$ ($j = 1,2$) are entire functions of the first order [2] and each of them is an even function of ξ; therefore, according to Hadamard's theorem the Weierstrass representation for each of them has the form

$$F_j(\xi) = d_j\prod_{m=1}^{\infty}(1 - \xi^2/a_{mj}^2),$$

where d_j is a constant and a_{mj} are zeros of the function $F_j(\xi)$ ($m = 1,2,\ldots,\infty$). Using the expressions for $F_j(\xi)$ we readily obtain

$$d_j = \lim_{\xi\to 0}F_j(\xi) = \lim F_j^+(\xi) = F_j^+(0) = \beta_j \quad (j = 1,2).$$

As a result, according to the residue theory, for $x > 0$ ($\mathrm{Re}\,\xi < 0$), we obtain

$$T_j^*(x,y) = \frac{q}{\lambda_j\beta_j}\sum_{m=1}^{\infty}\Big\{\Big[\alpha_j a_{m0}^-\cos\Big(a_{m0}^-(y-y_j)\Big) + \beta_j\sin\Big(a_{m0}^-(y-y_j)\Big)\Big]X\Big(a_{m0}^-\Big)\Big/\Big(a_{m0}^-\Big)^2e^{a_{m0}^- x}\Big\}.$$

Here

$$X(y) = \prod_{m=1}^{\infty}\Big(1 - y/a_{mk}^-\Big)\Big/\Big(\prod_{m=1}^{\infty}{}'\Big(1 - y/a_{mj}^+\Big)\prod_{m=1}^{\infty}{}'\Big(1 - y/a_{m0}^-\Big)\Big);$$

a prime indicates that terms in the products are dropped if they are equal to zero; $a_{mj}^{\pm}$ are zeros of the functions $F_j(\xi)$ ($j = 0,1,2$ and $m = 1,2,\ldots,\infty$) lying on the right-hand and left-hand complex half-planes, respectively.

For $x < 0$ ($\mathrm{Re}\,\xi > 0$) we have

$$T_j^*(x,y) = -\frac{q}{\lambda_j\beta_j}\Big\{\sum_{m=1}^{\infty}\Big[\alpha_j a_{mj}^+\cos\Big(a_{mj}^+(y-y_j)\Big) + \beta_j\sin\Big(a_{mj}^+(y-y_j)\Big)\Big]X(a_{mj}^+)\Big/(a_{mj}^+)^2e^{a_{mj}^+x} + \beta_j(y-y_j) + \alpha_j\Big\}.$$

So far we have assumed that $q = \text{const}$. If q appearing in formulas (6) has the form

$$q = \lambda_1\partial T_2^{(0)}/\partial y\Big|_{y=0} = \lambda_2\partial T_2^{(0)}/\partial y\Big|_{y=0} = q_0e^{p_n x},$$

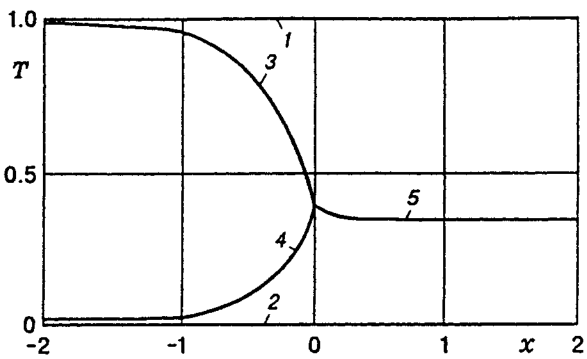

Fig. 1.

where $p_n > 0$ $(x < 0)$, it is sufficient to set $A_1^*(\xi) = a/\left(F^-(\xi)(\xi - p_n)\right)$. Taking into account that the function $A_1^*(\xi)$ now has an additional pole at the point $\xi = p_n$, after analogous calculations ($p_n \neq a_{mj}$; $n, m = 1, 2, \ldots$; $j = 0, 1, 2$), we find

$$T_j^*(x,y) = \frac{q}{\lambda_j\beta_j}\left\{\sum_{m=1}^{\infty}\left[\alpha_j a_{m0}^- \cos\left(a_{m0}^-(y-y_j)\right) + \beta_j \sin\left(a_{m0}^-(y-y_j)\right)\right] X(a_{m0}^-) e^{a_{m0}^- x} \Big/ \left(a_{m0}^-\right)^2\right\} \quad (x > 0); \quad (22)$$

$$T_j^*(x,y) = -\frac{q}{\lambda_j\beta_j}\left\{\sum_{m=1}^{\infty}\left[\alpha_j a_{mj}^+ \cos\left(a_{mj}^+(y-y_j)\right) + \beta_j \sin\left(a_{mj}^+(y-y_j)\right)\right] X(a_{mj}^+) e^{a_{mj}^+ x} \Big/ \left(\left(a_{mj}^+\right)^2 \left(a_{mj}^+ - p_n\right)\right)\right.$$

$$\left. + \left[\alpha_j p_n \cos(p_n(y-y_j)) + \beta_j \sin(p_n(y-y_j))\right] X(p_n) e^{-p_n x} \big/ p_n^2 + \beta_j(y-y_j) + \alpha_j\right\} \quad (x < 0). \qquad (23)$$

Since any function $f(t)$ that is continuous on the interval $[0, 1]$ can be approximated with any degree of accuracy by a polynomial of the form $Q_N(t) = \sum_{n=0}^{N} q_n t^{p_n}$ (t^{p_n} is a complete set of functions in the interval $[0, 1]$ and p_n are real), introducing new variable $t = e^x$ $(x < 0)$, we write the function $q(x)$ as

$$q(x) = q(\ln t) = q_*(t) = \sum_{k=0}^{\infty} q_k t^{p_k} = \sum_{k=0}^{\infty} q_k e^{p_k x}.$$

The solution then is represented by a superposition of solutions (22) and (23). The function $q(x)$ is not a constant if γ_j are functions of x. Then it is sufficient to apply a Laplace transform with respect to the x coordinate to determine $T_j^{(0)}(x,y)$ and $q(x) = \lambda_1 \partial T^{(0)}/\partial y\big|_{y=0}$. Summing solutions for (22) and (23), we obtain the desired solution for $q(x) = \sum_{k=0}^{\infty} q_k e^{p_k x}$. Figure 1 shows the temperature $T_j(x,y)$ at the point $y = 0$ as a function of x on different sides of the boundary for the case of $\alpha_j = 0$, $\beta_j = 1$ $(j = 1, 2)$, $\gamma_1 = -1$, $\gamma_2 = 0$, $\gamma = 1$, $h_1 = 1$, $h_2 = 2$ (curves 1 and 2 represent the temperature distribution at the external surfaces, curves 3 and 4, on the sides of the slit, and curve 5, in the ideal contact region).

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